Dimensionality Selection of Hyperbolic Graph Embeddings using Decomposed Normalized Maximum Likelihood Code-Length

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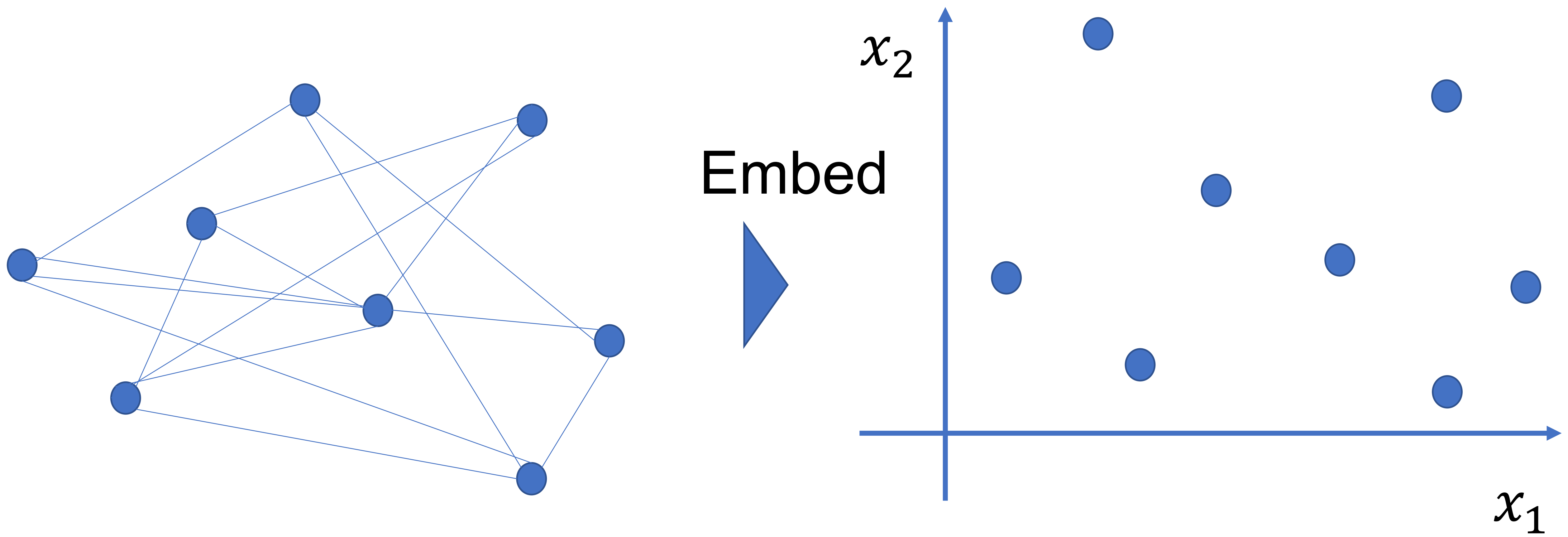
Agenda

1. Background

Graph Embeddings and Dimensionality Selection

Convert discrete representation to continuous one.



- Node classification [1].
- Link prediction [2].
- Clustering [3].
- Visualization [4].

Dimensionality controls

1.performance (e.g., underfitting with low dimensionality and overfitting with high dimensionality).

2.time and space computational complexity.

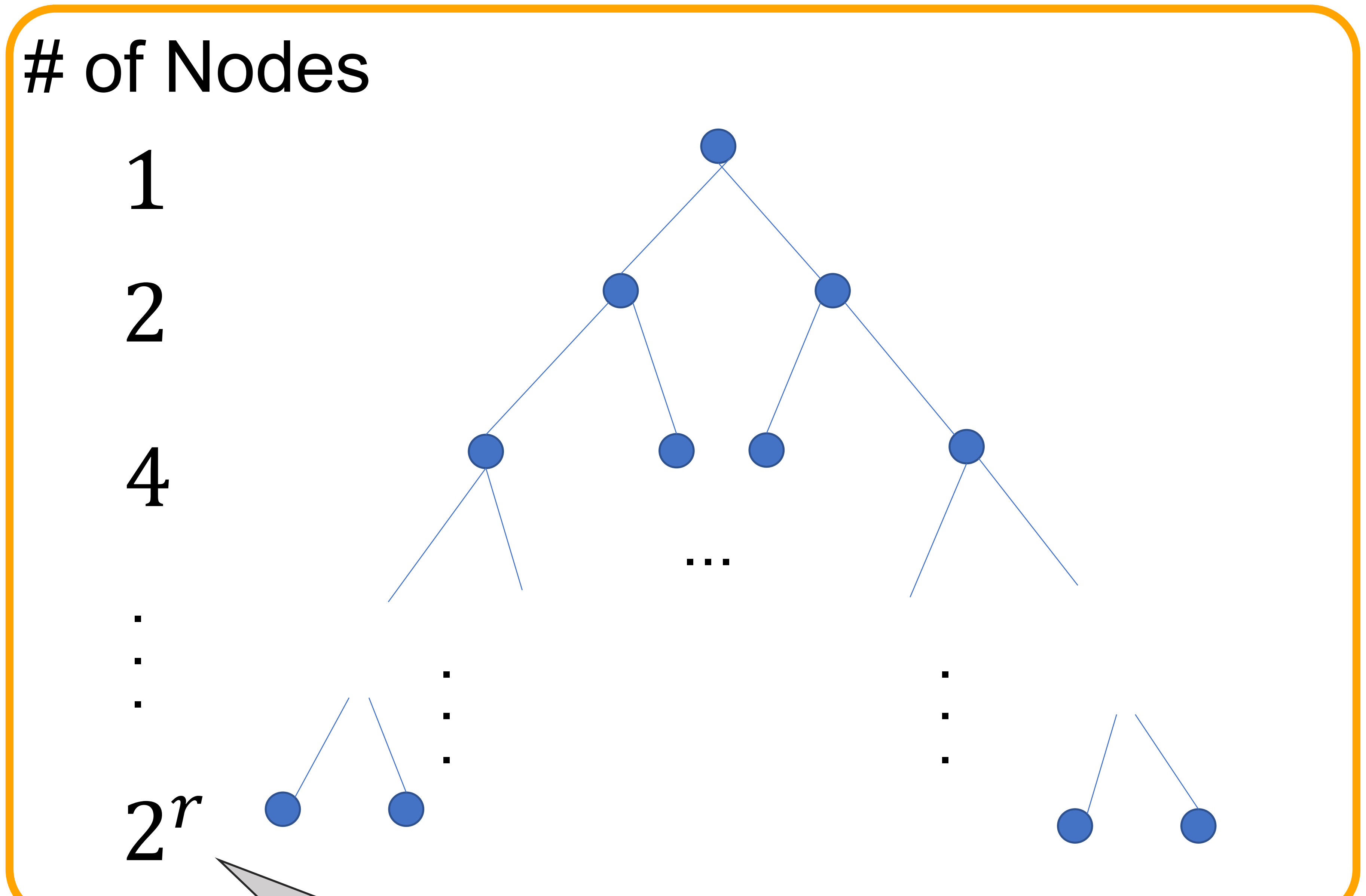
Our contribution: proposed dimensionality selection method for hyperbolic graph embeddings.

Hyperbolic Embeddings

Effective on hierarchical or tree-like structured graphs.

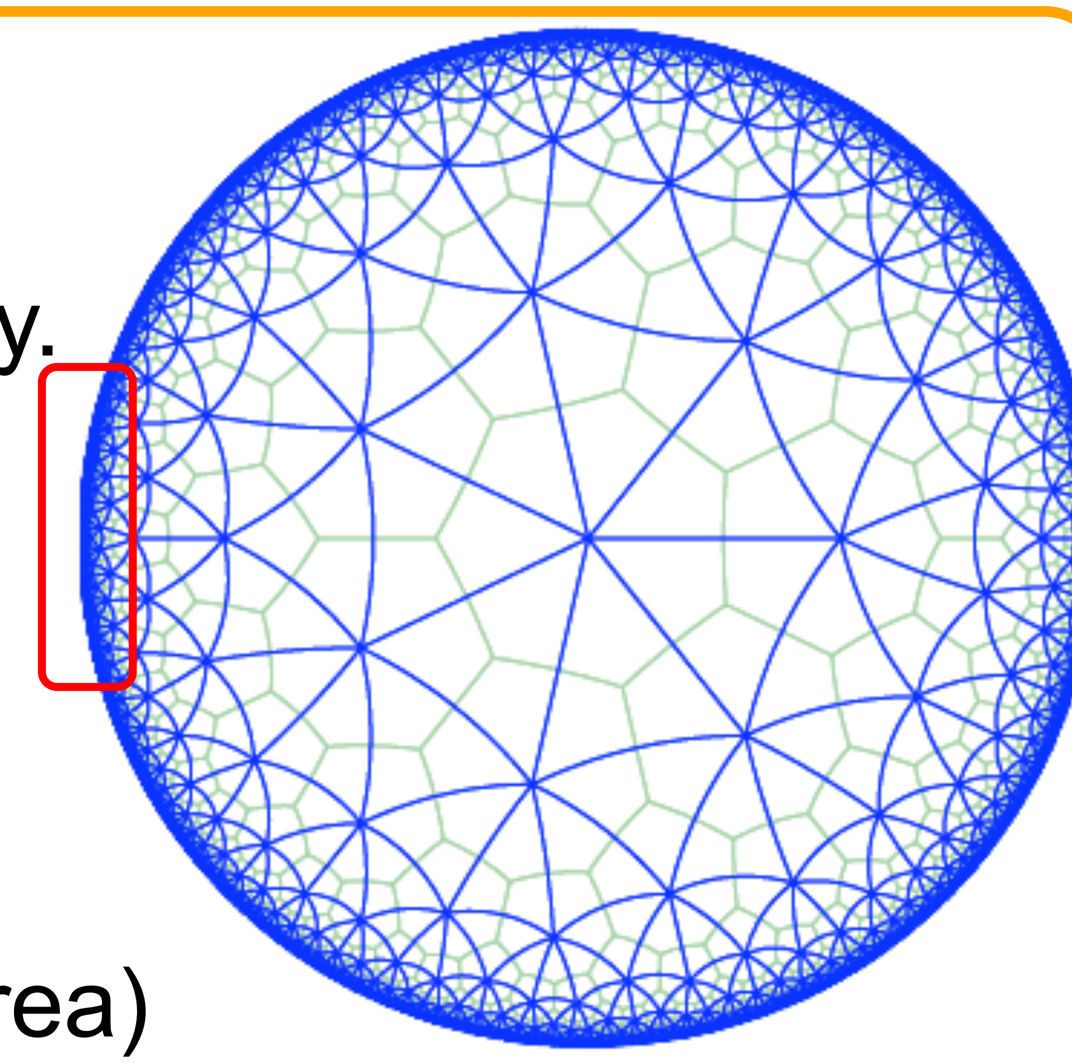
- The performance in **5-dimensional** hyperbolic space is better than that of **200-dimensional** Euclidean space for several graph mining tasks [8].

Tree



Hyperbolic Space

Vaster near the boundary.



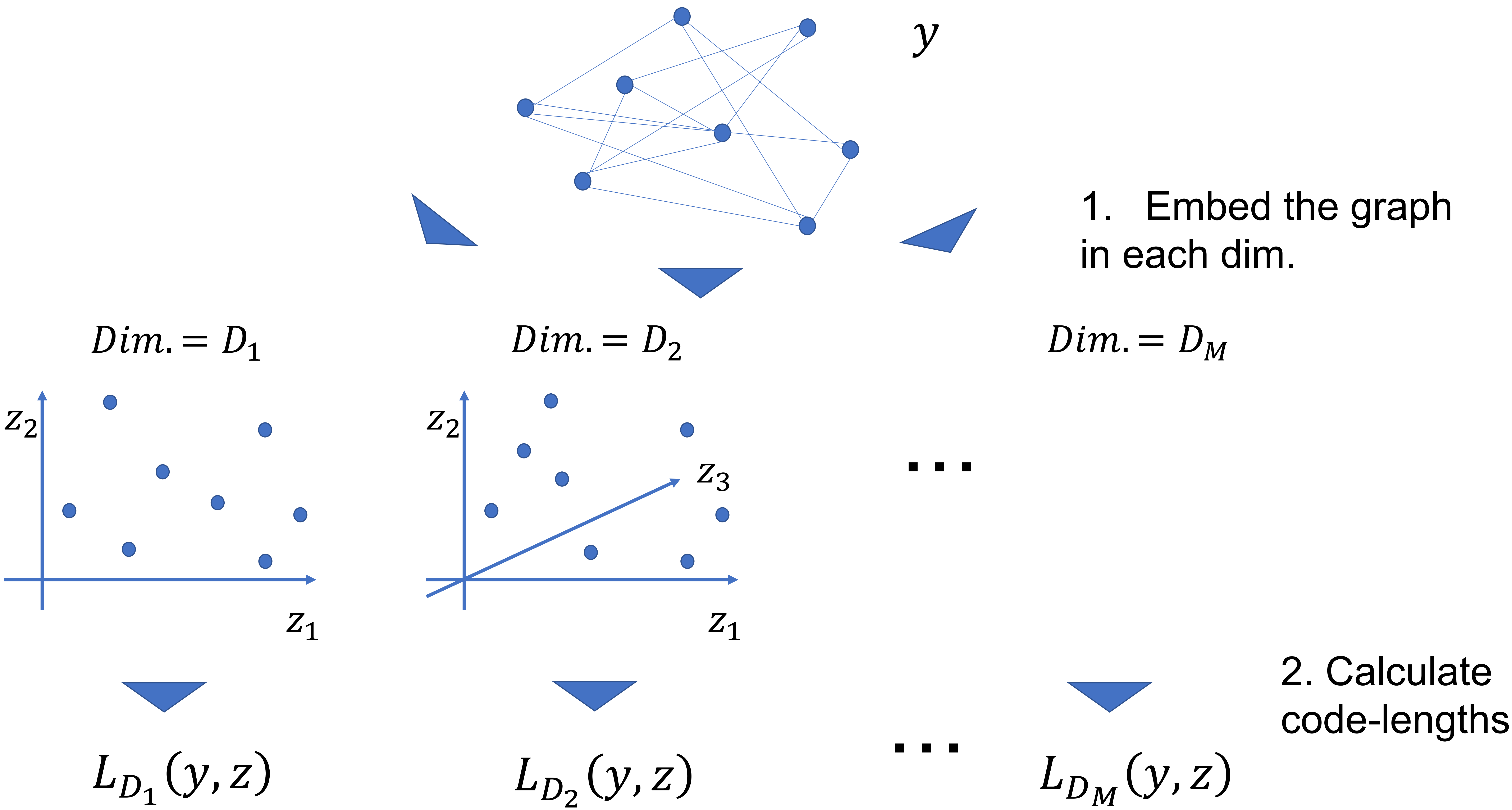
$\mathcal{H}^d$: (surface area) $\propto e^{(d-1)r}$

Fig.1 Triangles in a hyperbolic space (from [9]).

Exponential $\leftarrow$ **Analogy !** $\rightarrow$ **Exponential**

But the surface area in $\mathcal{R}^d$ is proportional to r^{d-1}. $\rightarrow$ **Polynomial !**

Proposed Method



→ Select the dimensionality that minimizes $L_{D_k}(y,z)$.

Agenda

2. Dimensionality Selection using DNML Code-Length

MDL Principle [10]

Select the model that minimizes the code-length.

- One of the information criteria (e.g., AIC [11], BIC [12], etc).
- Theoretical properties such as consistency in model selection [10], etc.

- To apply the MDL principle, we need to do the following:

1. Formulate hyperbolic embeddings as a probabilistic model.

2. Derive $L_D(y, z)$: the encoding function associated with dimensionality $D \in \mathcal{M}$.

1. Latent Variable Model (LVM)

Use LVM with hyperbolic geometric graphs (HGGs).

- Latent variable model (hyperbolic geometric graph, HGG) [9, 13]:

$p(y, z) = p(y|z)p(z)$.

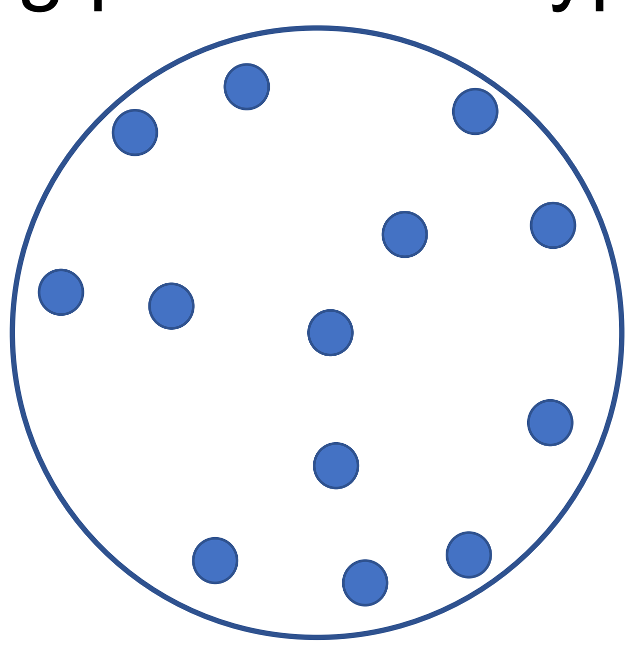
$z = \{z_i\}_{i \in [n]}$: embedding as latent variables.

$y = \{y_{ij}\}_{(i,j) \in \Lambda_{[n]}}$: observed edges.

Properties

- Power law of degree distribution [14].
- High clustering coefficient [14].

1. Sampling points in a hyperbolic space.



$$p(z; \sigma, R) = \prod_{i \in [n]} p(z_i; \sigma, R),$$

$$p(z_i; \sigma, R) = p(r_i; \sigma, R) \prod_{j=1}^{D-1} p(\theta_{ij}),$$

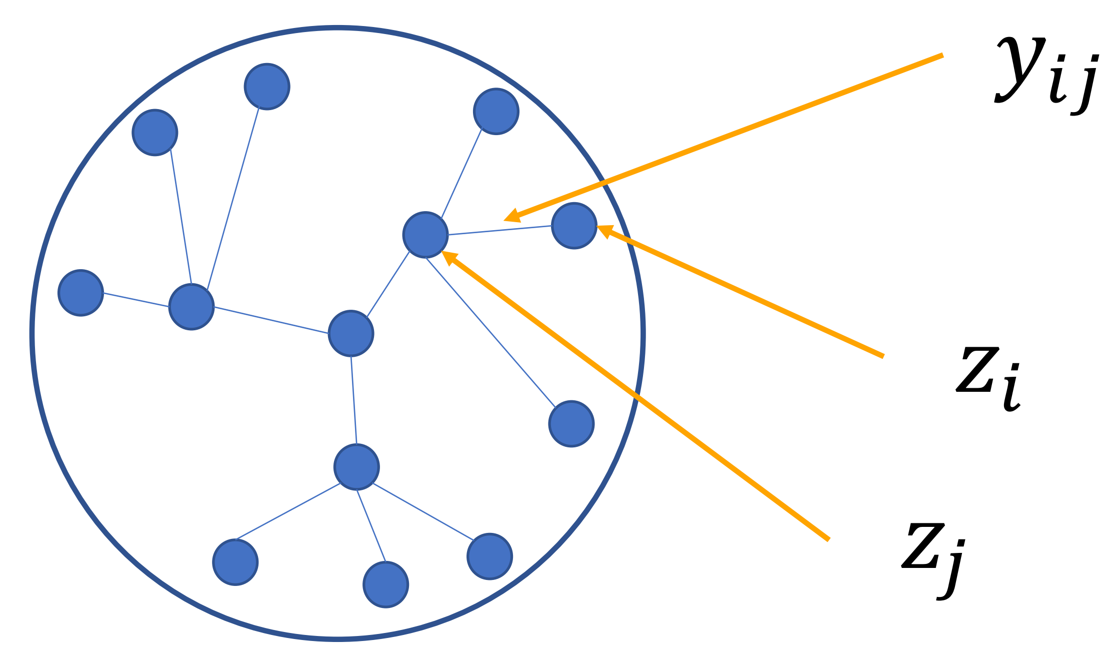
$$p(\theta_{ij}) = \begin{cases} \frac{\sin^{D-1-j} \theta_{ij}}{I_{D,j}} & (j \in [D-2]), \\ \frac{1}{2\pi} & (j = D-1), \end{cases}$$

$$p(r_i; \sigma, R) = \frac{\sinh^{D-1}(\sigma r_i)}{C_D(\sigma)},$$

where $I_{D,j} = \int_0^\pi \sin^{D-1-j} \theta d\theta$ and $C_D(\sigma) = \int_0^R \sinh^{D-1}(\sigma r) dr$.

Pseudo-uniform dist. of hyperbolic

2. Connect points with some probability.



$$p(\boldsymbol{y}; \boldsymbol{\phi}, \beta) = \prod_{(i,j) \in \Lambda_{[n]}} p(y_{ij}; \phi_i, \phi_j, \beta),$$

$$p(y_{ij}; \phi_i, \phi_j, \beta) = \begin{cases} \frac{1}{1+\exp(\beta(d_{\phi_i \phi_j} - R))} & (y_{ij} = 1), \\ 1 - \frac{1}{1+\exp(\beta(d_{\phi_i \phi_j} - R))} & (y_{ij} = 0). \end{cases}$$

Sigmoid function.

2. Decomposed Normalized Maximum Likelihood (DNML) Code-Length [15, 16]

Use DNML code-length for LVM.

- DNML is easy to calculate for latent variable models:

$$L_{\mathrm{DNML}}(\boldsymbol{y}, z) := L_{\mathrm{NML}}(\boldsymbol{y}|z) + L_{\mathrm{NML}}(z),$$

where NML code-lengths [17] are given by

$$L_{\mathrm{NML}}(\boldsymbol{y}|z) := \boxed{-\log p(\boldsymbol{y}|z; \hat{\beta}(\boldsymbol{y}, z))} + \boxed{\log \sum_{\boldsymbol{y}'} p(\boldsymbol{y}'|z; \hat{\beta}(\boldsymbol{y}', z))},$$

Negative logarithm of the maximum likelihood

Penalty term (parametric complexity).

$$L_{\mathrm{NML}}(z) := \boxed{-\log p(z; \hat{\sigma}(z))} + \boxed{\log \int dz' p(z'; \hat{\sigma}(z'))}.$$

Our contribution: derived the explicit form of the approximation of each penalty term.

Approximation of DNML #1

NML is approximated with Fisher information [17, 18].

$$L_{\mathrm{NML}}(\boldsymbol{y}|\boldsymbol{z}) \approx -\log p(\boldsymbol{y}|\boldsymbol{z};\hat{\beta}(\boldsymbol{y},\boldsymbol{z})) + \frac{1}{2}\log\frac{n(n-1)}{4\pi} + \underline{\log\int_{\beta_{\min}}^{\beta_{\max}}\sqrt{|I_n(\beta)|}\,d\beta},$$

Integral over the parameter domain.

$$L_{\mathrm{NML}}(\boldsymbol{z}) \approx -\log p(\boldsymbol{z};\hat{\sigma}(\boldsymbol{z})) + \frac{1}{2}\log\frac{n}{2\pi} + \underline{\log\int_{\sigma_{\min}}^{\sigma_{\max}}\sqrt{|I(\sigma)|}\,d\sigma},$$

Integral over the parameter domain.

$$I(\sigma) := \lim_{n\to\infty}\frac{1}{n}E_{\sigma}\left[-\frac{\partial^2\log p(z;\sigma)}{\partial\sigma^2}\right] = E_{\sigma}\left[-\frac{\partial^2\log p(z;\sigma)}{\partial\sigma^2}\right].$$

$$I_n(\beta) := E_{\beta}\left[-\frac{2}{n(n-1)}\frac{\partial^2}{\partial\beta^2}\sum_{(i,j)\in\Lambda_{[n]}}\log p(y_{ij}|z_i,z_j;\beta)\right].$$

Fisher information

Approximation of DNML #2

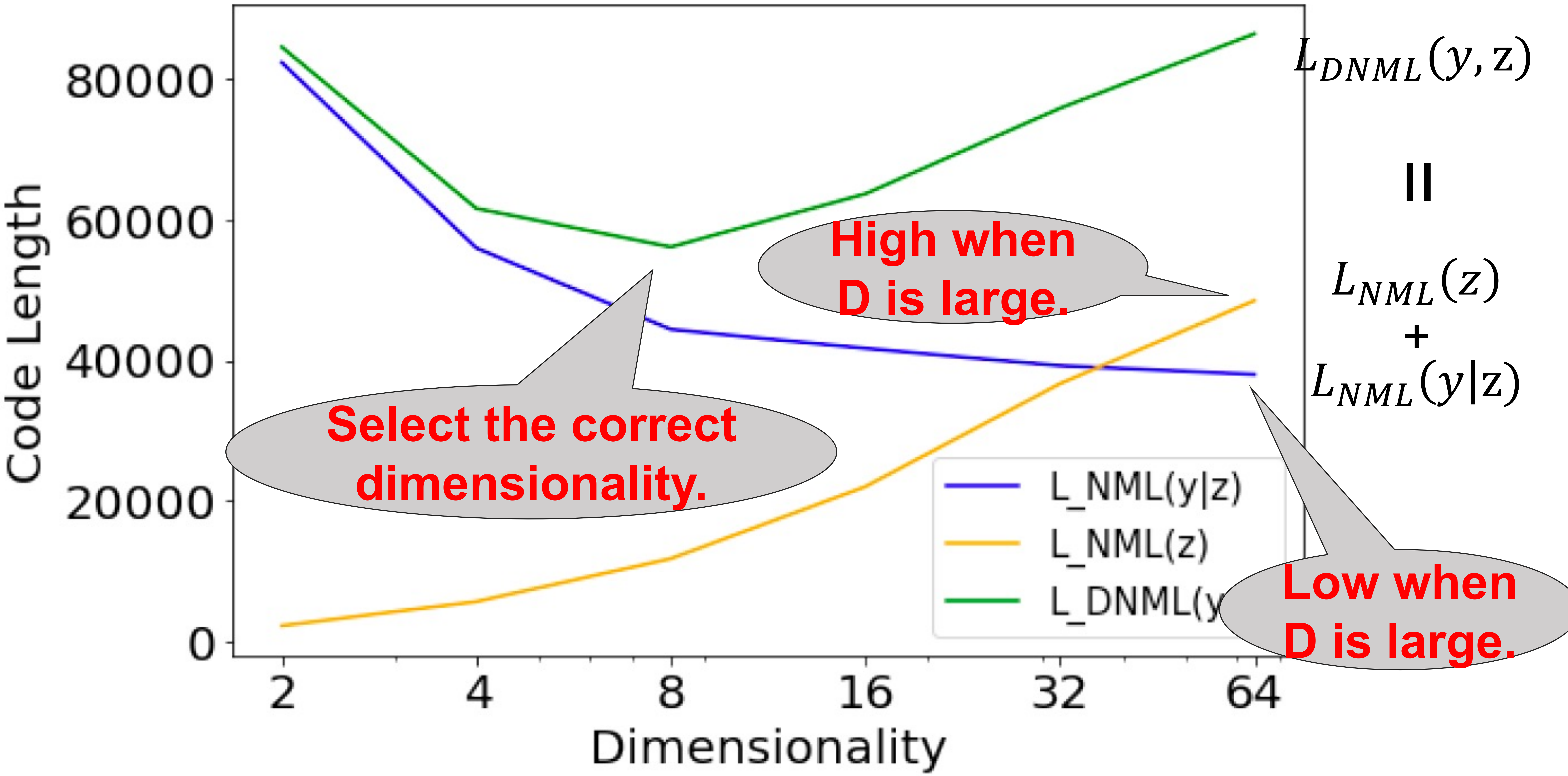
- With some calculation, the Fisher information are given by:

$$I_n(\beta) = \frac{2}{n(n-1)} \sum_{(i,j)\in\Lambda_{[n]}} \frac{(R - d_{z_i z_j})^2 \exp(-\beta(R - d_{z_i z_j}))}{\left(1 + \exp(-\beta(R - d_{z_i z_j}))\right)^2},$$

$$I(\sigma) = (D-1)^2 \frac{\int_0^R r^2 \cosh^2(\sigma r) \sinh^{D-3}(\sigma r) dr}{C_D(\sigma)} + \left\{ \frac{\int_0^R (D-1) r' \cosh(\sigma r') \sinh^{D-2}(\sigma r') dr'}{C_D(\sigma)} \right\}^2.$$

- Numerical integration with the Gaussian quadrature [19].

Typical Behavior of DNML



$$L_{DNML}(y, z) = L_{NML}(z) + L_{NML}(y|z)$$

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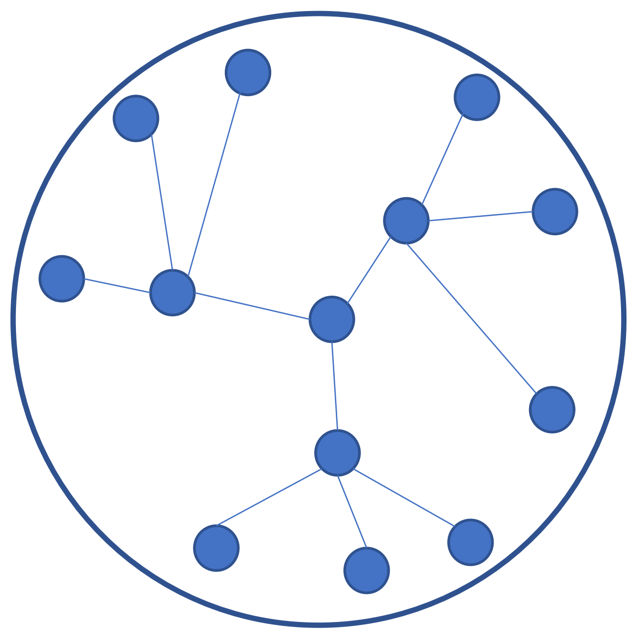
3. Experimental Results

1. Artificial Dataset

DNML-HGG selected the correct dimensionality with enough data.

Setting

1. Generate artificial graphs with $D_{true} = 16$.



2. Estimate $\widehat{D}$ with the proposed method and the following competitive methods:

- AIC [11]
- BIC [12]
- MinGE [20]
  (Euclidean dimensionality selection method)

Results

Table 1 Average benefits on HGG-16.

# of nodes	DNML-HGG	AIC	BIC	MinGE
400	**0.042**	0.000	0.000	0.000
800	**0.250**	0.000	0.000	0.000
1600	**0.042**	0.000	0.000	0.000
3200	0.000	**0.250**	0.000	0.000
6400	**1.000**	0.375	0.000	0.000
12800	**1.000**	0.542	0.000	0.000

Best !

$$b(\hat{D}, D_{\mathrm{true}}) = \max\left\{0, 1 - \frac{|\log_2 \hat{D} - \log_2 D_{\mathrm{true}}|}{T_{\mathrm{gap}}}\right\},$$

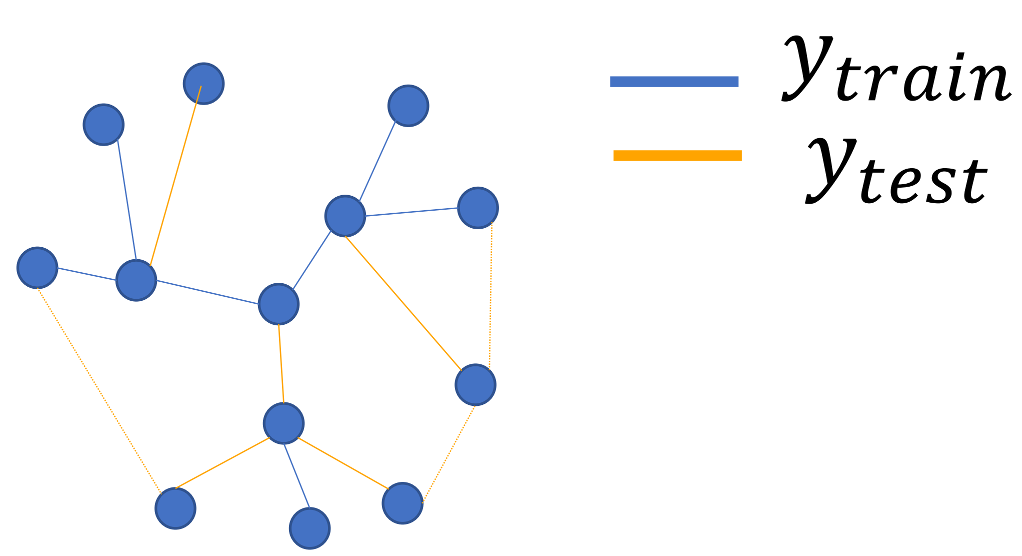
where $T_{gap} = 2, D_{true} = 16.$

2. Scientific Collaboration Networks

The selected dimensionality suppresses the computational resource, whereas keeping the AUC.

Setting

1. Separate y into y_{train} and y_{test}.



2. Train with y_{train} and estimate $\hat{D}$.
Then, perform link prediction on y_{test}.

Results

Selected Dim. Max. Performance

AUC (Four graphs)

Network	2	4	8	16	32	64
AstroPh	0.549	0.813	0.842	0.846	0.849	**0.850**
CondMat	0.522	0.754	**0.767**	0.764	0.766	**0.767**
GrQc	0.526	0.728	0.744	**0.756**	0.755	**0.756**
HepPh	0.559	0.847	0.880	0.880	0.883	**0.885**

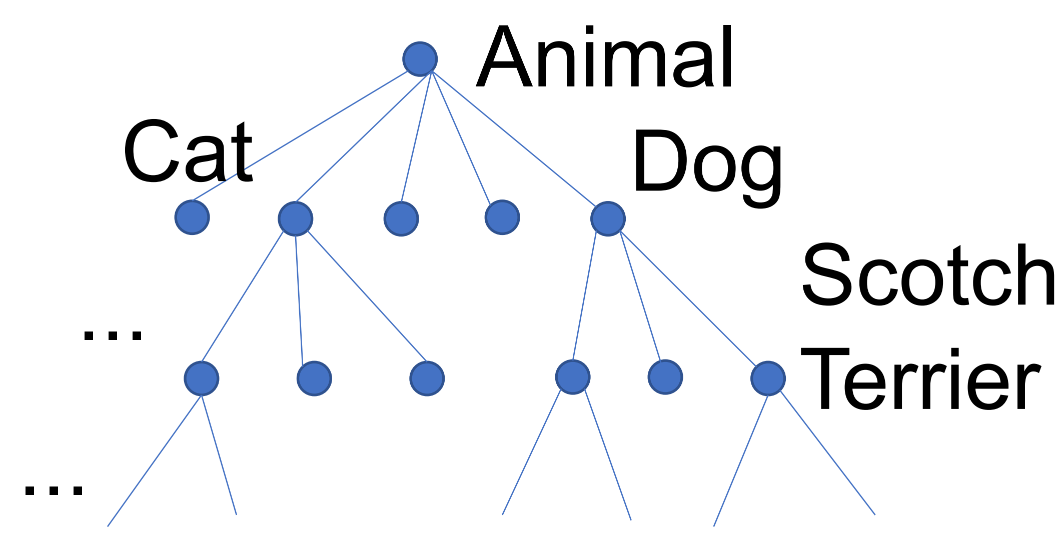
≥(Maximum)-0.01

3. WordNet[21] Datasets

DNML-HGG selected the dimensionality that preserves the hierarchy of the graphs.

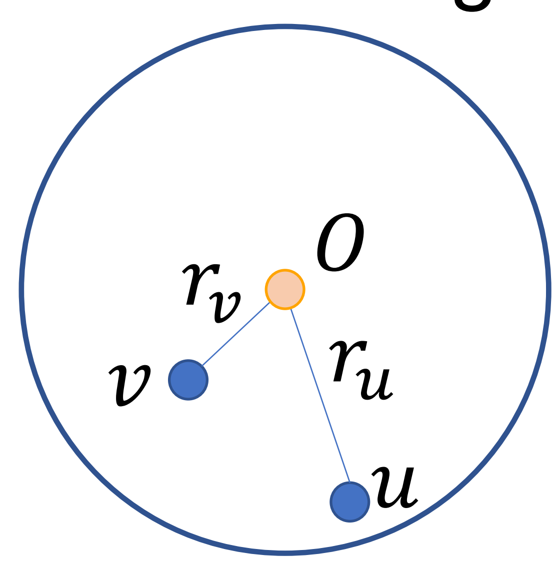
Setting

1. Generate graphs with hierarchical structure using is-a relation.



2. Calculate is-a-scores.

Embedding



$$\text{is-a-score}(\boldsymbol{u}, \boldsymbol{v}) = (\alpha(r_u - r_v) - 1) d_{\boldsymbol{uv}},$$

→High when "u is a v".

Results

Benefit
(Average of six graphs)

DNML-HGG	AIC	BIC	MinGE
0.714	0.286	0.500	0.000

Best !

$$b(\hat{D}, D_{\text{true}}) = \max\left\{0, 1 - \frac{|\log_2 \hat{D} - \log_2 D_{\text{true}}|}{T_{\text{gap}}}\right\},$$

where $T_{gap} = 2, D_{true} = \max_D is - a - score.$

Agenda

4. Summary
&
Future Perspectives

Summary & Future Perspective

- **Summary**
  - Contribution #1: proposed dimensionality selection method for hyperbolic graph embeddings.
    - Few studies for hyperbolic embeddings [22].
  - Contribution #2: derived the explicit form of the approximation of DNML.
  - Experimentally showed the effectiveness of the proposed method.

- **Future Perspective**
  - Extension for other Riemannian manifolds.
    - Euclidean spaces, spherical spaces, etc…

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Appendix: Hyperboloid Model [23]

Hyperbolic Space:

$$\mathbb{H}^D := \{\boldsymbol{x} = (x_0, x_1, \ldots, x_D)^\top \mid \boldsymbol{x} \in \mathbb{R}^{D+1}, < x, x >_{\mathcal{L}} = -1, x_0 > 0\}$$

where $\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{\mathcal{L}} = \boldsymbol{u}^\top g_D \boldsymbol{v}.$ $g_D = \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \in \mathbb{R}^{(D+1)\times(D+1)},$

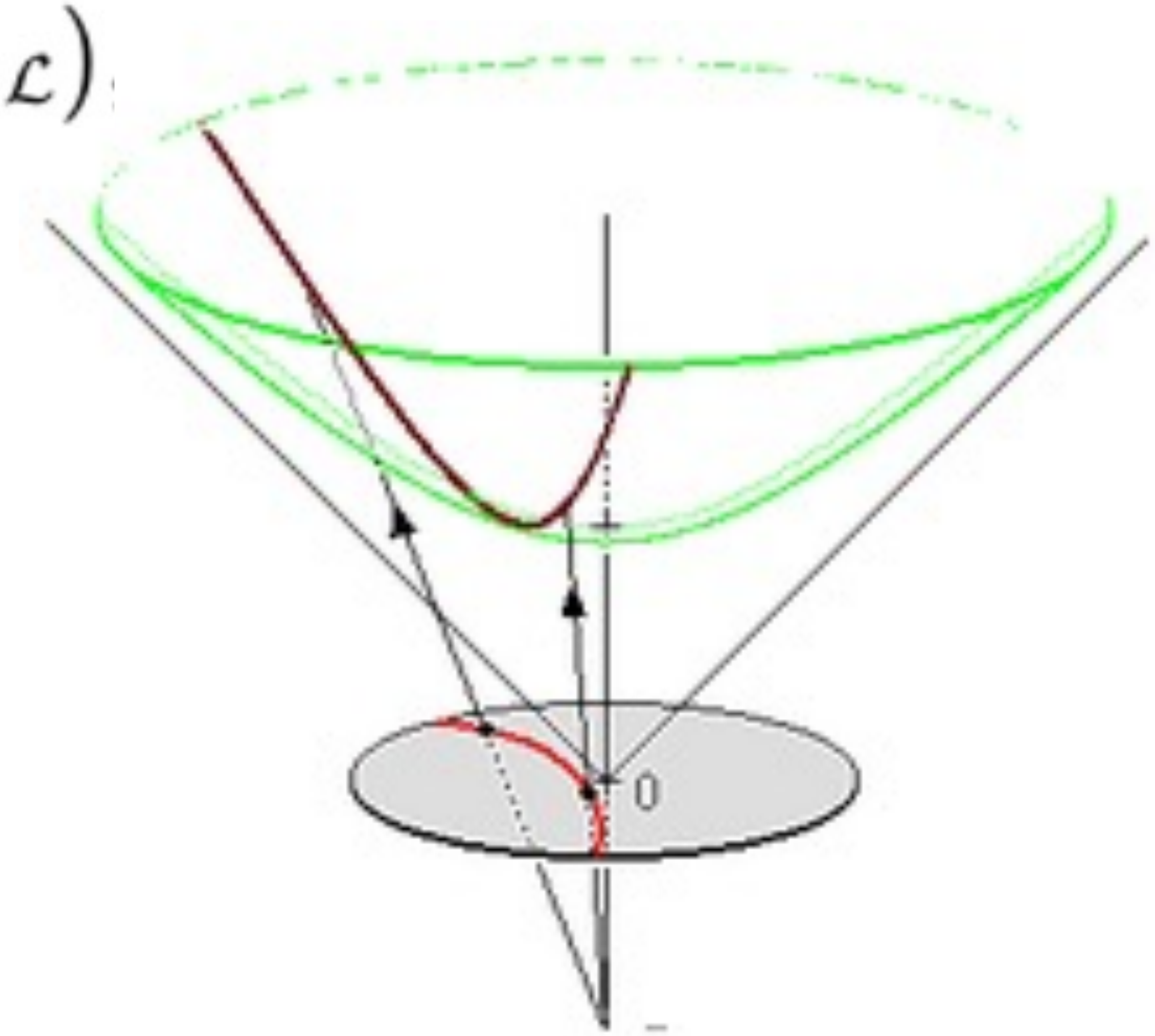
Distance: $d_{\boldsymbol{uv}} = \operatorname{arcosh}(- < \boldsymbol{u}, \boldsymbol{v} >_{\mathcal{L}})$

Exponential Map:

$$\operatorname{Exp}_{\boldsymbol{z}}(\boldsymbol{u}) := \cosh(\sqrt{\langle \boldsymbol{v}, \boldsymbol{v} \rangle_{\mathcal{L}}})\boldsymbol{z} + \sinh(\sqrt{\langle \boldsymbol{v}, \boldsymbol{v} \rangle_{\mathcal{L}}})\frac{\boldsymbol{v}}{\sqrt{\langle \boldsymbol{v}, \boldsymbol{v} \rangle_{\mathcal{L}}}},$$

Merits

- Numerically stable.
- Exponential map is easy to implement.



*https://en.wikipedia.org/wiki/Hyperboloid_model

Appendix: Loss Function

Optimize $-\log p(\boldsymbol{y}, \boldsymbol{z}; \boldsymbol{\beta}, \boldsymbol{\sigma})$ through stochastic framework [24].

$$
\begin{aligned}
L(z, \beta, \sigma) &= -\log p(y|z;\beta) - \log p(z;\sigma) \\
&= \underbrace{\sum_{(i,j)\in\Lambda_{[n]}} -\log p(y_{ij}|z_i, z_j;\beta)}_{\propto n^2} + \underbrace{\sum_{i\in[n]} -\log p(z_i;\sigma)}_{\propto n}
\end{aligned}
$$

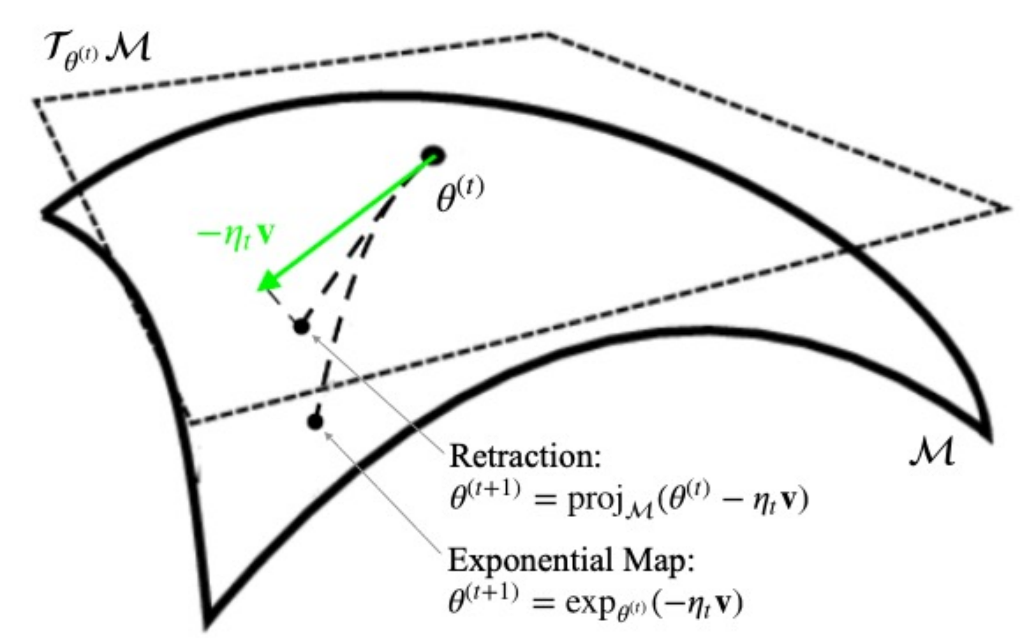
→Different sample size.

$$
= \sum_{(i,j)\in\Lambda_{[n]}} \left\{ -\log p(y_{ij}|z_i, z_j;\beta) - \frac{1}{n-1}\log p(z_i;\sigma) - \frac{1}{n-1}\log p(z_j;\sigma) \right\}.
$$

Summation over all possible pairs.

Division by n-1.

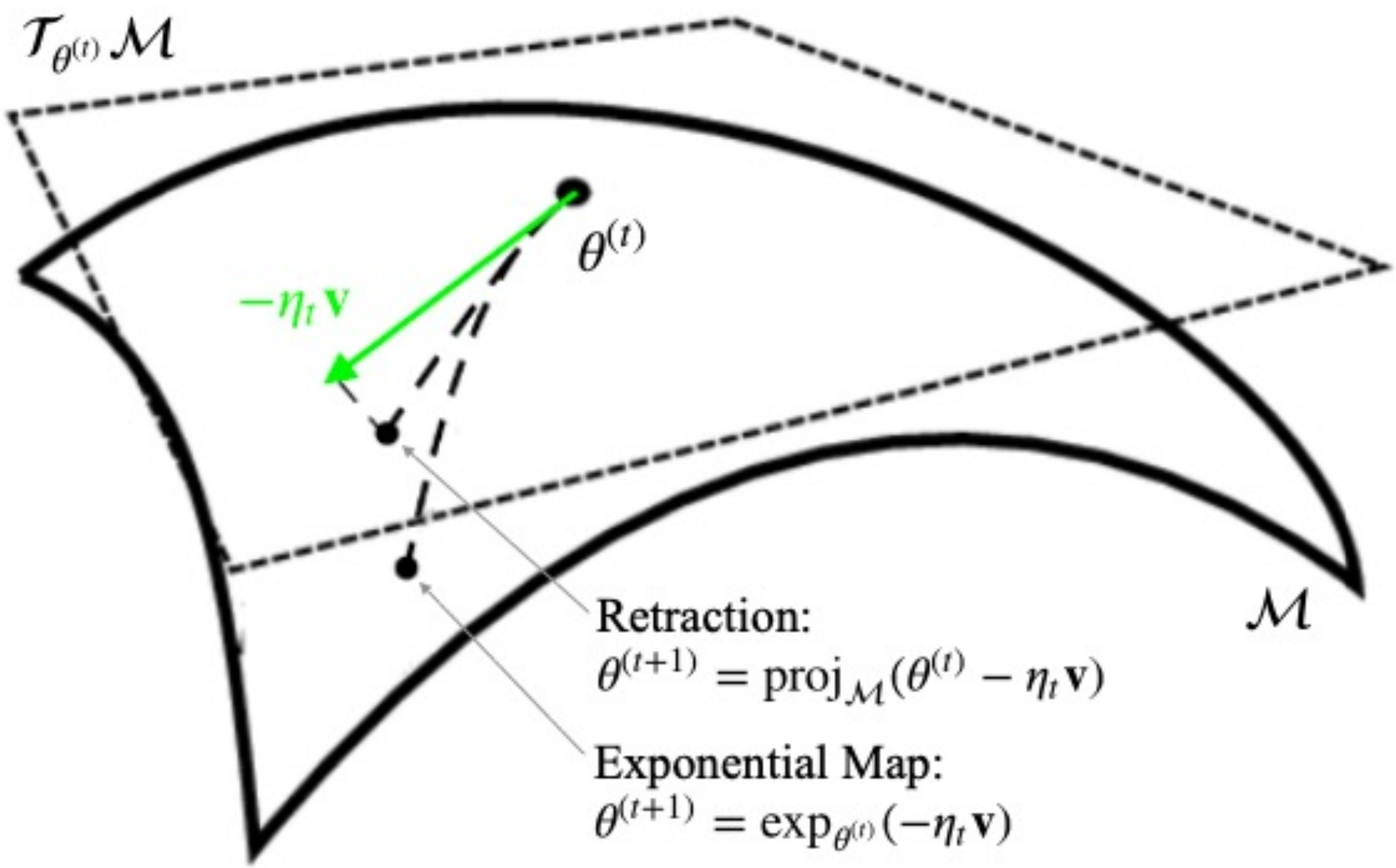
→Uniformly sample S ⊂(possible pairs).
Riemannian Gradient Descent [25].



*https://andbloch.github.io/Stochastic-Gradient-Descent-on-Riemannian-Manifolds/

Appendix: Riemannian Gradient Descent

Use Riemannian Gradient Descent [25].



*https://andbloch.github.io/Stochastic-Gradient-Descent-on-Riemannian-Manifolds/

Appendix: Derivation of DNML #1

For $x_{ij} := R - d_{z_i z_j} \in [-R, R]$, we reformulate $p(y_{ij}|z_i, z_j; \beta)$ as follows:

$$
\begin{aligned}
p(y_{ij}|z_i, z_j; \beta) &= p(y_{ij}|x_{ij}; \beta), \\
&= p_{ij}^{y_{ij}} (1 - p_{ij})^{(1-y_{ij})},
\end{aligned}
$$

where $p_{ij} = 1/(1+\exp(-\beta x_{ij}))$. This form is the logistic regression with the constraint $\beta \in [0, \beta_{\max}]$. Then, the negative logarithm of the likelihood $L(\beta)$ is

$$
\begin{aligned}
L(\beta) &:= - \sum_{(i,j)\in\Lambda_{[n]}} \log p(y_{ij}|x_{ij}; \beta), \\
&= - \sum_{(i,j)\in\Lambda_{[n]}} \left\{ y_{ij} \log \frac{p_{ij}}{1 - p_{ij}} + \log(1 - p_{ij}) \right\}, \\
&= - \sum_{(i,j)\in\Lambda_{[n]}} \left\{ y_{ij} \beta x_{ij} + \log(1 + \exp(\beta x_{ij})) \right\}.
\end{aligned}
$$

Appendix: Derivation of DNML #2

Hence, we obtain

$$\frac{\partial L(\beta)}{\partial \beta} = -\sum_{(i,j)\in\Lambda_{[n]}} \left\{ y_{ij}x_{ij} - \frac{x_{ij}}{1+\exp(-\beta x_{ij})} \right\},$$

$$\frac{\partial^2 L(\beta)}{\partial \beta^2} = \sum_{(i,j)\in\Lambda_{[n]}} \frac{x_{ij}^2 \exp(-\beta x_{ij})}{\left(1+\exp(-\beta x_{ij})\right)^2}.$$

Since $\partial^2 L(\beta)/\partial\beta^2$ is independent of y_{ij}, we derive

$$I_n(\beta) = \frac{2}{n(n-1)} \sum_{(i,j)\in\Lambda_{[n]}} \frac{x_{ij}^2 \exp(-\beta x_{ij})}{\left(1+\exp(-\beta x_{ij})\right)^2}.$$

Appendix: Derivation of DNML #3

The negative logarithm of the likelihood for $z = (r, \theta_1, \ldots, \theta_{D-1})$ is

$$L(\sigma) := -\log \frac{\sinh^{D-1}(\sigma r)}{C_D(\sigma)} - \sum_{j=1}^{D-2} \log \frac{\sin^{D-1-j}\theta_j}{I_{D,j}} - \log \frac{1}{2\pi}.$$

Interchanging the derivative and the integral, we obtain

$$\frac{\partial L(\sigma)}{\partial \sigma}$$
$$= -(D-1)\frac{r\cosh \sigma r}{\sinh \sigma r} + \frac{\int_0^R (D-1) r' \cosh(\sigma r') \sinh^{D-2}(\sigma r') dr'}{C_D(\sigma)}.$$

Similarly, we get

$$\frac{\partial^2 L(\sigma)}{\partial \sigma^2} = (D-1)\frac{r^2}{\sinh^2(\sigma r)}$$
$$+(D-1)\frac{\int_0^R r'^2 \sinh^{D-1}(\sigma r') + (D-2) r'^2 \cosh^2(\sigma r') \sinh^{D-3}(\sigma r') dr'}{C_D(\sigma)}$$
$$-\left\{\frac{\int_0^R (D-1) r' \cosh(\sigma r') \sinh^{D-2}(\sigma r') dr'}{C_D(\sigma)}\right\}^2,$$

Appendix: Derivation of DNML #4

Note that the second and third terms are independent of r. The expectation of the first term with respect to r is calculated as

$$(D-1)\int_0^R \frac{\sinh^{D-1}(\sigma r)}{C_D(\sigma)} \cdot \frac{r^2}{\sinh^2(\sigma r)} dr = \frac{D-1}{C_D(\sigma)} \int_0^R r^2 \sinh^{D-3}(\sigma r) dr.$$

Finally, we derive the following:

$$I(\sigma) = (D-1)^2 \frac{\int_0^R r^2 \cosh^2(\sigma r) \sinh^{D-3}(\sigma r) dr}{C_D(\sigma)} - \left\{ \frac{\int_0^R (D-1) r' \cosh(\sigma r') \sinh^{D-2}(\sigma r') dr'}{C_D(\sigma)} \right\}^2.$$

Appendix: Competitive Methods

- Three methods were chosen:

$$\mathrm{AIC}(\boldsymbol{y}, z; D) := -\log p(\boldsymbol{y}|z; \hat{\beta}(\boldsymbol{y}, z), \hat{\sigma}(z)) + (nD + 1),$$

$$\mathrm{BIC}(\boldsymbol{y}, z; D) := -\log p(\boldsymbol{y}|z; \hat{\beta}(\boldsymbol{y}, z), \hat{\sigma}(z)) + \frac{nD+1}{2} \log \frac{n(n-1)}{2}.$$

- AIC and BIC do not guarantee their rationales.
- Minimum graph entropy (MinGE [20]) is the dimensionality selection method of Euclidean embeddings.

Appendix: Non-identifiability Problem

- Lack of one-to-one correspondence between parameter and probability distribution.

- Conventional information criteria such as Akaike's information criterion (AIC) [11], Bayesian information criterion (BIC) [12], etc… do not guarantee their rationales because their derivation depend on the central limit theorem.

$$AIC = -\log p\left(x; \hat{\theta}\right) + k,$$

$$BIC = -\log p\left(x; \hat{\theta}\right) + \frac{k}{2} \log n,$$

where k is the number of free parameters, and n is the number of data.

Appendix: Non-identifiability Problem of Hyperbolic Graph Embeddings

$\phi_i \coloneqq (r_i, \theta_i)$: embeddings, $y \in \{0,1\}$: edges.

LEMMA 1. *Assume that $r_i \neq 0$ for some $i \in [n]$. For $\alpha \in (0, 2\pi)$, we define $\phi'_i \coloneqq (r_i, \theta_i + \alpha)$ for $i \in [n]$.* Rotation.

Then, $\phi'_i \in \mathbb{H}^D_R$ and $\boldsymbol{\phi} \neq \boldsymbol{\phi}' = \{\phi'_i\}_{i \in [n]}$. Moreover, the following equation holds:

$$p(\boldsymbol{y}; \boldsymbol{\phi}, \beta) = p(\boldsymbol{y}; \boldsymbol{\phi}', \beta).$$

The same at
two different parameters !

Therefore, the probability distribution of a Poincaré embedding is non-identifiable.

$$p(\boldsymbol{y}; \boldsymbol{\phi}, \beta) = \prod_{(i,j) \in \Lambda_{[n]}} p(y_{ij}; \phi_i, \phi_j, \beta),$$

$$p(y_{ij}; \phi_i, \phi_j, \beta) = \begin{cases} \frac{1}{1+\exp(\beta(d_{\phi_i \phi_j} - R))} & (y_{ij} = 1), \\ 1 - \frac{1}{1+\exp(\beta(d_{\phi_i \phi_j} - R))} & (y_{ij} = 0). \end{cases}$$

Sigmoid function.

Appendix: Training Detail

- All embeddings were initialized uniformly at random over $[-0.001, 0.001]^D$.
- Chose the following parameters:
  - $\sigma_{max} = 1.0, \sigma_{min} = 0.001,$
  - $\beta_{max} = 10.0, \beta_{min} = 1.0,$
  - $\beta^{(0)} = 1.0, \sigma^{(0)} = 1.0,$
  - $R = \log n$.
- When making mini-batches, 10 negative samples were sampled per a positive sample.
- The learning rate was 0.1 for the first 10 epochs, and 34.375 for the remaining 790 epochs.
- The number of epochs was 800.
- The likelihood and the Fisher information were approximated as follows:

$$-\log p(\boldsymbol{y}|z;\hat{\beta}(\boldsymbol{y},z)) \approx -\frac{|\boldsymbol{y}|}{|\boldsymbol{y}'_{\mathrm{train}}|}\log p(\boldsymbol{y}'_{\mathrm{train}}|z;\hat{\beta}(\boldsymbol{y}_{\mathrm{train}},z)),$$

$$I_n(\beta) \approx \frac{2}{n'(n'-1)}\sum_{(i,j)\in\Lambda'_S}\frac{(R-d_{z_iz_j})^2\exp(-\beta(R-d_{z_iz_j}))}{(1+\exp(-\beta(R-d_{z_iz_j})))^2},$$

where $S \subset [n]$ was sampled uniformly at random over $[n]$, and $\Lambda'_S := \{(i,j)|(i,j)\in S\times S, i<j\}$.

Appendix: Selected Dimensionalities on Artificial Datasets

# of Nodes	DNML-HGG	AIC	BIC	MinGE
400	4.3±1.1	4.0±0.0	3.5±0.87	64.0±0.0
800	6.0±2.0	4.0±0.0	4.0±0.0	64.0±0.0
1600	4.3±1.1	4.0±0.0	4.0±0.0	64.0±0.0
3200	4.0±0.0	6.0±2.0	4.0±0.0	64.0±0.0
6400	16.0±0.0	7.0±1.7	4.0±0.0	64.0±0.0
12800	16.0±0.0	8.7±2.2	4.0±0.0	64.0±0.0

Correct dim. with enough nodes

Underestimate

Overestimate

Appendix: Selected Dimensionalities on Scientific Collaboration Networks

Table 5 Selected dimensionalities of each method.

Network	DNML-HGG	AIC	BIC	MinGE
AstroPh	16	16	4	64
CondMat	16	16	4	64
GrQc	16	8	4	64
HepPh	16	8	4	64

Appendix: Consiceness

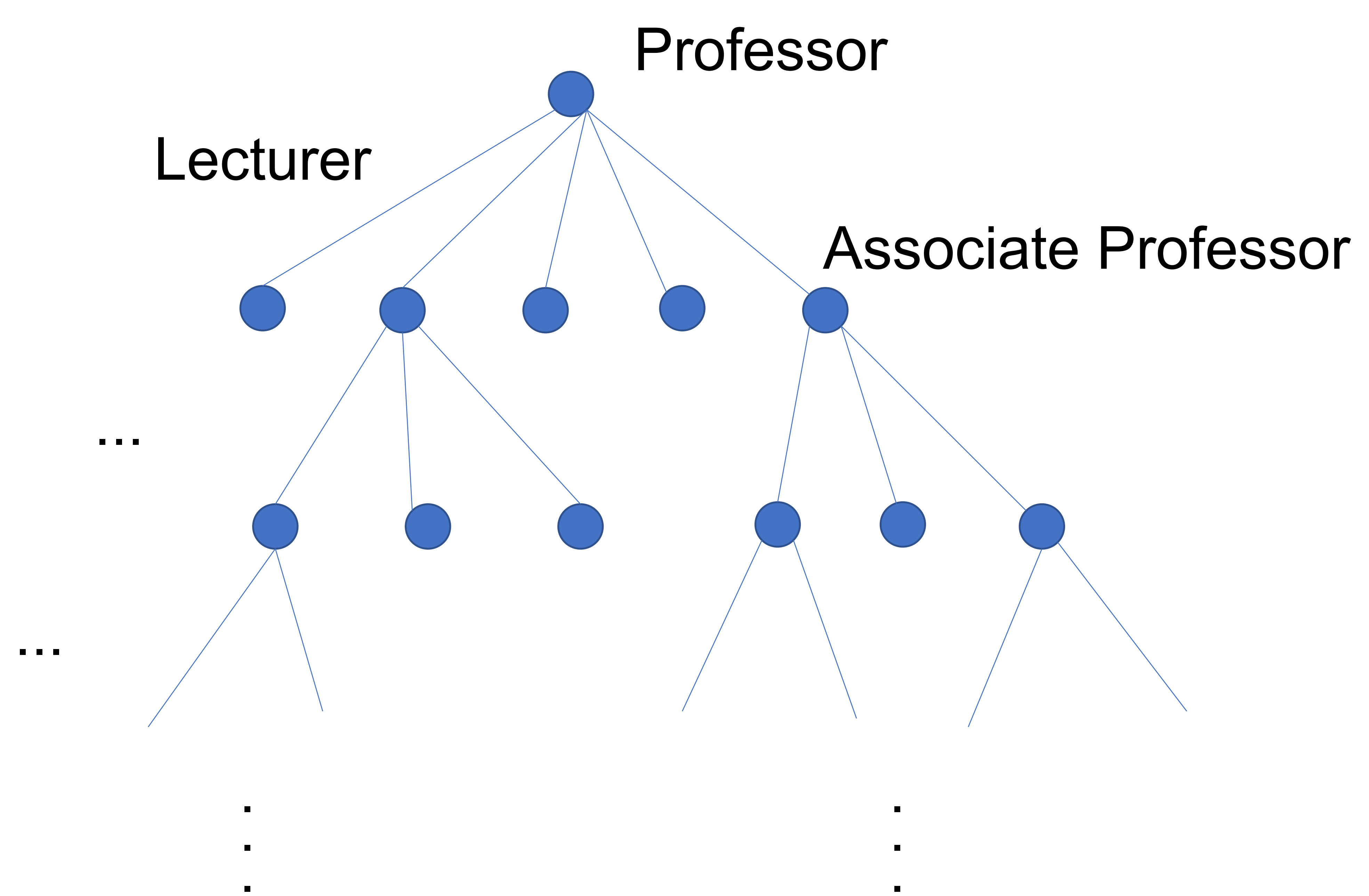
Table 6 Average conciseness of each method with $\epsilon_{\max} = 0.01$.

Network	DNML-HGG	AIC	BIC	MinGE
AstroPh	**0.501**	0.378	0.000	0.000
CondMat	**0.451**	**0.473**	0.000	0.000
GrQc	**0.944**	**0.948**	0.080	0.000
HepPh	**0.367**	0.176	0.000	0.000

Appendix: Hierarchical Structure of Scientific Collaboration Networks

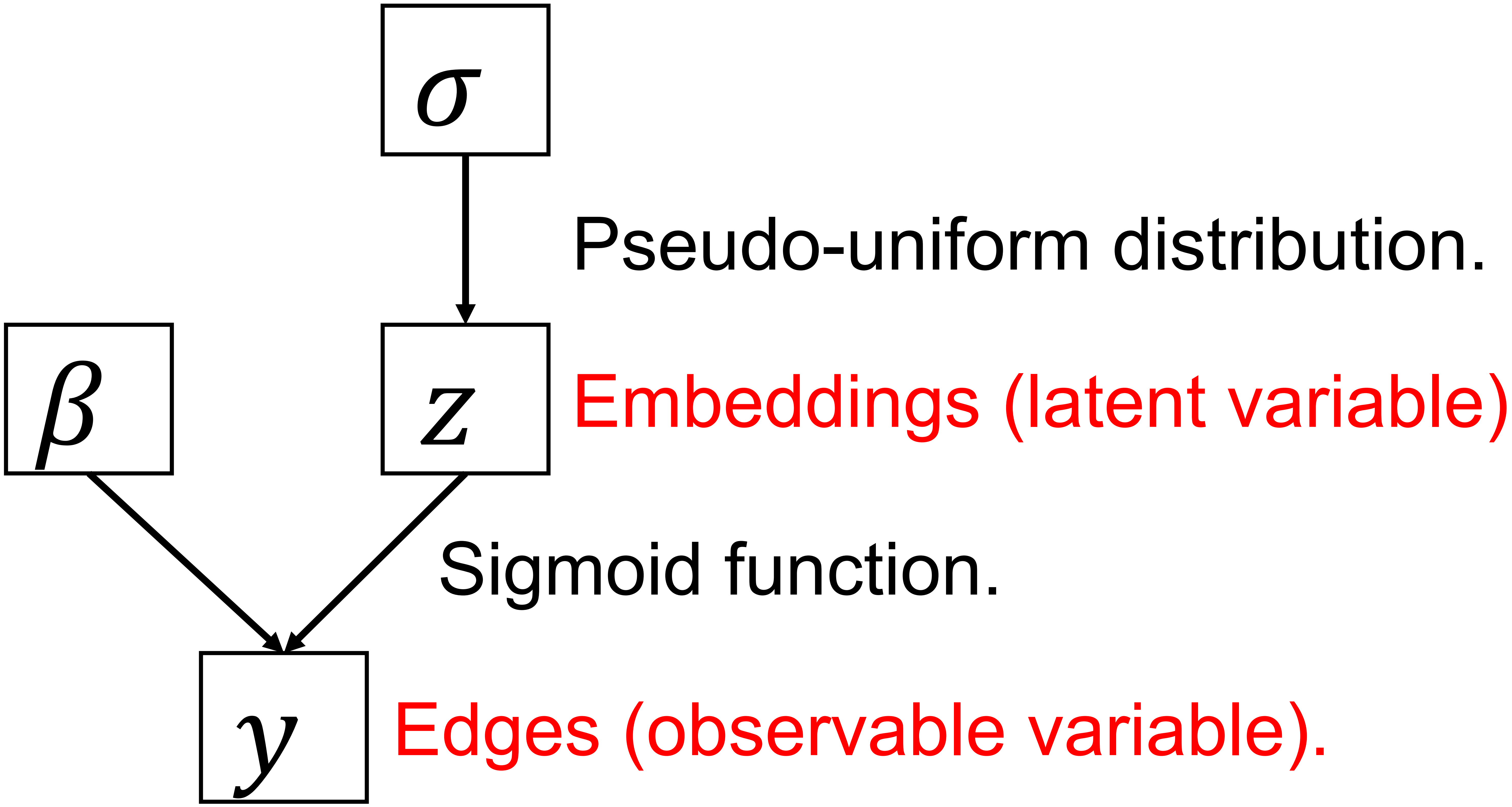
Real-world complex networks tend to have hierarchical structures.

- Ex: networks of co-authorship, paper citation, web pages, etc.



- We can approximate such structures as trees.

Appendix: Graphical Model



Appendix: Power law of degree distributions

- $P(k) \propto k^{-\gamma}$: degree distribution (k is the degree).
  - The majority of papers are cited infrequently, while a small number of papers are cited frequently.

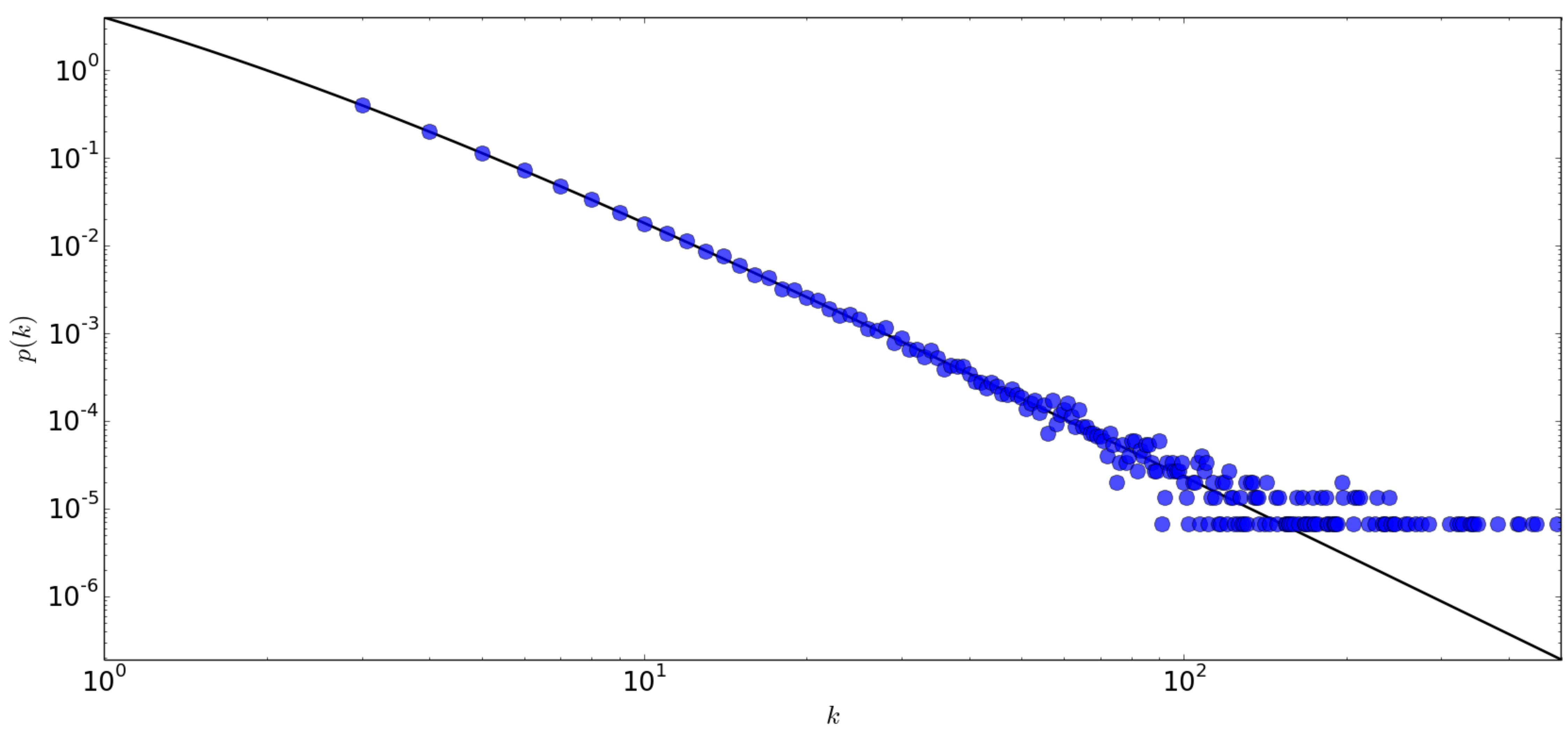
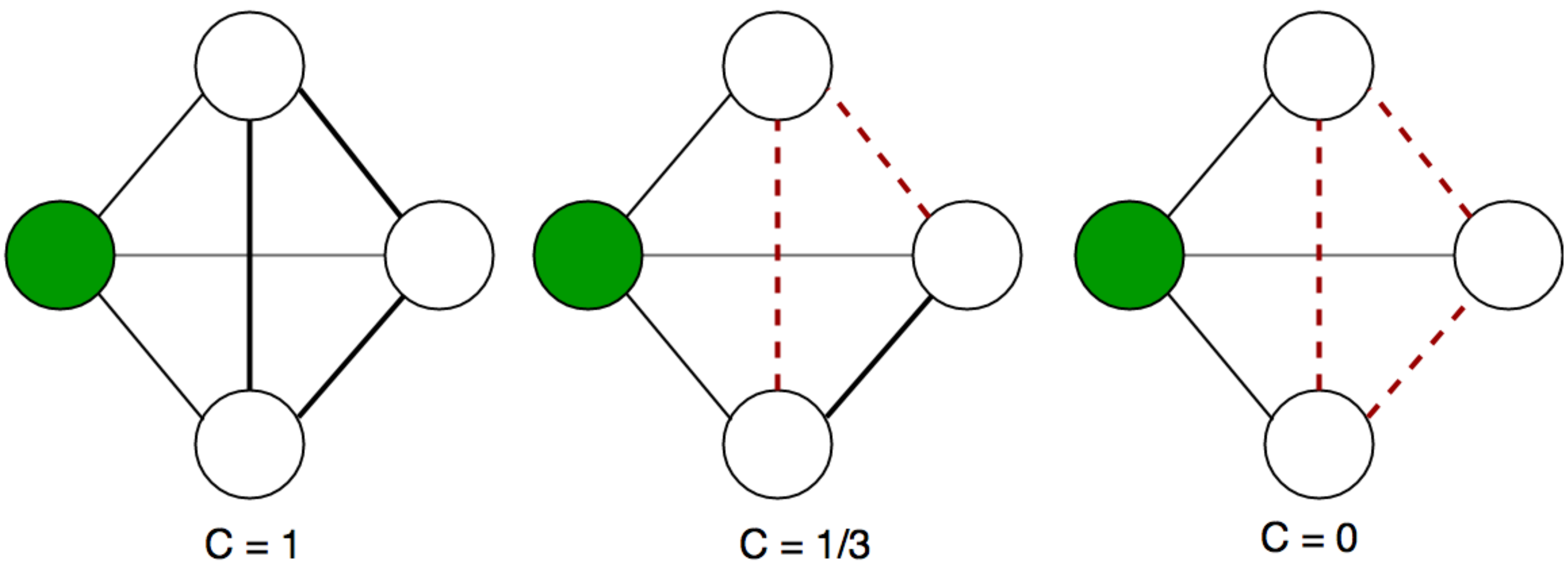


FIg2. degree distribution in a graph (https://en.wikipedia.org/wiki/Scale-free_network).

Appendix: High Clustering Coefficient

High when the graph contains many triangles.



*https://www.geeksforgeeks.org/clustering-coefficient-graph-theory/

Appendix:Ordinal MDL Principle

- Normalized maximum likelihood (NML, [10]) code length is defined as:
- $L_{NML}(x^n) = -\log p_{NML}(x^n)$
$$= \underbrace{-\log p\left(x^n; \hat{\theta}(x^n)\right)}_{} + \underbrace{\log \sum_{x'^n \in \mathcal{X}^n} p\left(x'^n; \hat{\theta}(x'^n)\right)}_{}$$

Low when the model is complex. High when the model is complex.

where $\mathcal{X}$ is the data domain and $\hat{\theta}(x)$ is the maximum likelihood estimate.